Reusing Information for Multifidelity Active Learning in Reliability-Based Design Optimization

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This paper develops a multifidelity method to reuse information from optimization history for adaptively refining surrogates in reliability-based design optimization (RBDO). RBDO can be computationally prohibitive due to numerous evaluations of the expensive high-fidelity models to estimate the probability of failure of the system in each optimization iteration. In this work, the high-fidelity model evaluations are replaced by cheaper-to-evaluate adaptively refined surrogate evaluations in the probability of failure estimation. The method reuses the past optimization iterations as an information source for devising an efficient multifidelity active learning (adaptive sampling) algorithm to refine the surrogates that best locate the failure boundary. We implement the information-reuse method using a multifidelity extension of efficient global reliability analysis that combines the expected feasibility function with a weighted lookahead information gain criterion to pick both the next sample location and information source used to evaluate the sample.

I. Introduction

Accounting for uncertainties during the design process is key to obtaining engineering systems that satisfy performance requirements. Traditionally, safety factors have been used to compensate for uncertainties in a deterministic optimization setup. However, in order to design more efficient and safe systems, deterministic optimization is being replaced by optimization under uncertainty (OUU). OUU is a two-loop process consisting of the outer-loop optimization and the inner-loop uncertainty propagation process as shown in Figure 1(a). The focus of this paper is on reliability-based design optimization (RBDO) [1–4], where the inner-loop uncertainty propagation involves reliability analysis in each optimization iteration. The reliability analysis requires estimating a probability of failure, which is typically done using Monte Carlo sampling for nonlinear systems. When the underlying system models are expensive to evaluate, RBDO can quickly become computationally prohibitive.

Using cheap-to-evaluate approximations (surrogates) of the high-fidelity models in the reliability estimation is one way of reducing the computational cost as shown in Figure 1(b) [5–9]. We focus on Monte Carlo methods for reliability analysis because of their wide applicability to nonlinear systems. In order to get efficient and accurate estimates of probability of failure through Monte Carlo methods, we require accurate predictions from the surrogate in the neighborhood of the failure boundary, which is defined by a specific contour of the limit state function. This can be done by adaptively refining the surrogate around the failure boundary. In this paper, we achieve this using a multifidelity active learning method that reuses existing information.

Adaptive refinement of surrogates around the failure boundary has been addressed in the literature through single-fidelity adaptive sampling algorithms using only a single high-fidelity model. Such methods fall primarily into two groups, using either support vector machines (SVM) or Gaussian process (GP) surrogates. Adaptive SVM methods have been applied to reliability analysis and contour location [10–12]. The GP-based (sometimes referred to as kriging-based) methods use the GP prediction mean and prediction variance to devise acquisition functions for greedy and lookahead adaptive sampling methods to refine the surrogates around the target region. Efficient Global Reliability Analysis (EGRA) adaptively refines GP surrogates around the failure boundary for reliability analysis [13, 14]. EGRA was also combined with efficient global optimization (a.k.a. Bayesian optimization) for RBDO [9]. Picheny et al. [15] used a weighted integrated mean square error criterion to adaptively refine the kriging surrogate to accurately approximate the failure boundary. Bect et al. [16] defined a one-step lookahead strategy for surrogate refinement called stepwise

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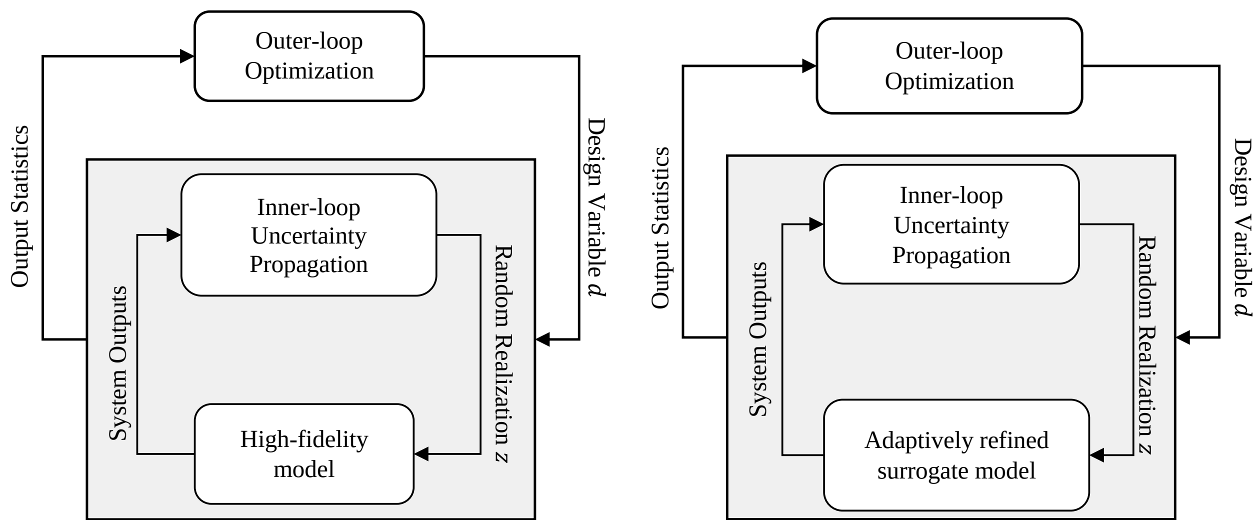


(a) Using high-fidelity model

(b) Using adaptively refined surrogate model

Fig. 1 Optimization under uncertainty using the high-fidelity model, or a cheaper adaptively refined surrogate model that best predicts the failure boundary.

uncertainty reduction for estimating probability of failure. Dubourg et al. [8] proposed refining kriging surrogates using a population-based adaptive sampling technique through subset simulation for RBDO. However, the above methods use only the high-fidelity model evaluations to refine the surrogate and the cost of building these adaptive surrogates can also become expensive.

Using multiple cheaper sources of information along with the high-fidelity model to build and refine the surrogates can further reduce the computational effort. The surrogates from all the past RBDO iterations are a natural source of readily available information and tend to be considerably cheaper to evaluate than the high-fidelity model. In this work, we develop a method to reuse that information along with the high-fidelity model to further reduce the cost of refining the surrogates around the failure boundary. Since we have multiple fidelities available through the information-reuse method, we require a multifidelity active learning method to utilize the different sources. There is limited literature on multifidelity active learning methods for refining a surrogate around the failure boundary. A hierarchical bi-fidelity adaptive SVM construction for locating limit state function contours was proposed by Dribusch et al. [17]. The recently proposed CLoVER (**C**ontour **Lo**cation **V**ia **E**ntropy **R**eduction) method is a multifidelity active learning algorithm that uses a one-step lookahead entropy-reduction-based adaptive sampling criterion [18]. In this work, we apply the information-reuse method for locating the failure boundary using a multifidelity extension of EGRA. The adaptive sampling method picks the next location using the expected feasibility function and the next information source using a lookahead information gain criterion. We use a multifidelity GP [19] as the surrogate for the limit state function that combines information from the available information sources. The adaptively refined surrogates can then be used for computationally efficient reliability analysis in each RBDO iteration and stored for future reuse. Any multifidelity active learning method for locating the failure boundary can be used for information reuse in RBDO and we will pursue comparing the information-reuse method for other available sampling criterion in our future work.

The remainder of this paper is organized as follows. Section II describes the RBDO problem formulation. Section III describes the method of defining an additional information source using the RBDO history. Section IV presents the multifidelity active learning method used to leverage RBDO history for efficiently refining surrogates around the failure boundary. Section V presents the results showing the efficiency of the method followed by conclusions in Section VI.

II. Problem setup

The inputs to the system are N_d design variables $\boldsymbol{d} \in \mathcal{D} \subseteq \mathbb{R}^{N_d}$ and N_z uncertain random variables $Z \in \Omega \subseteq \mathbb{R}^{N_z}$ with the probability density function π. Here, $\mathcal{D}$ denotes the design space and Ω denotes the random sample space. The vector of a realization of the random variables Z is denoted by $\boldsymbol{z}$. The RBDO formulation used in this work is given by

$$\begin{aligned} \min_{\boldsymbol{d}\in\mathcal{D}} \quad & \mathbb{E}_{\pi}[f(\boldsymbol{d}, Z)] \\ \text{subject to} \quad & \mathbb{P}(g(\boldsymbol{d}, Z) > 0) \leq p_{\mathrm{T}}, \end{aligned} \tag{1}$$

where $f : \mathcal{D} \times \Omega \mapsto \mathbb{R}$ is the objective function quantity of interest, $g : \mathcal{D} \times \Omega \mapsto \mathbb{R}$ is the limit state function, and p_{T} is the acceptable threshold on the probability of failure. In this work, without loss of generality, the failure of the system is defined as $g(\boldsymbol{d}, \boldsymbol{z}) > 0$, which defines the reliability of the system in RBDO. We note that the failure boundary is defined as the zero contour of the limit state function, $g(\boldsymbol{d}, \boldsymbol{z}) = 0$, and any other failure boundary can be reformulated into this form.

One way to estimate the probability of failure $p_{\mathrm{F}}(\boldsymbol{d}) = \mathbb{P}(g(\boldsymbol{d}, Z) > 0)$ for nonlinear systems is Monte Carlo simulation. At the current outer-loop optimization iteration t, the Monte Carlo estimator of the probability of failure $\widehat{p}_{\mathrm{F}}(\boldsymbol{d}_t)$ for design $\boldsymbol{d}_t$ is given by

$$\widehat{p}_{\mathrm{F}}(\boldsymbol{d}_t) = \frac{1}{m} \sum_{i=1}^{m} \mathbb{I}_{\mathcal{G}_t}(\boldsymbol{d}_t, \boldsymbol{z}_i), \tag{2}$$

where $\boldsymbol{z}_i$, $i = 1, \ldots, m$ are m samples from probability density π, $\mathcal{G}_t = \{\boldsymbol{z} \mid \boldsymbol{z} \in \Omega,\ g(\boldsymbol{d}_t, \boldsymbol{z}) > 0\}$ is the failure set, and $\mathbb{I}_{\mathcal{G}_t} : \mathcal{D} \times \Omega \to \{0, 1\}$ is the indicator function defined as

$$\mathbb{I}_{\mathcal{G}_t}(\boldsymbol{d}_t, \boldsymbol{z}) = \begin{cases} 1, & \boldsymbol{z} \in \mathcal{G}_t \\ 0, & \text{else.} \end{cases} \tag{3}$$

The probability of failure estimation, repeated at every optimization iteration t, requires many evaluations of the expensive-to-evaluate high-fidelity model g, which can make RBDO computationally prohibitive. The computational cost can be reduced by replacing the high-fidelity evaluations by evaluations from a cheap-to-evaluate surrogate. In this work, we build surrogates adaptively refined around the failure boundary in the random variable space for a given design at each outer-loop optimization iteration.

The goal of this work is to reuse the available surrogates from past RBDO iterations as an extra information source instead of only the high-fidelity model to efficiently build the adaptive surrogates. We use a multifidelity active learning criterion that utilizes the multiple information sources to refine the surrogate to accurately locate the failure boundary. A multifidelity surrogate model $\widehat{g}_t(l, Z)$ approximates observations from each information source l in the random variable space. The subscript t denotes that the surrogate is built for a given design $\boldsymbol{d}_t$ at every optimization iteration t. The multifidelity surrogate model can provide predictions for any information source l and random variable realization $\boldsymbol{z}$. The multifidelity surrogate model simultaneously approximates all the information sources while encoding the correlations between the different information sources. The high-fidelity information source is corresponds to the index $l = 0$. The high-fidelity surrogate model prediction is given by $\widehat{g}_t(0, Z)$, which is used for the prediction of failure. Next we describe the different information sources and the multifidelity active learning method used for refining the surrogates around the failure boundary.

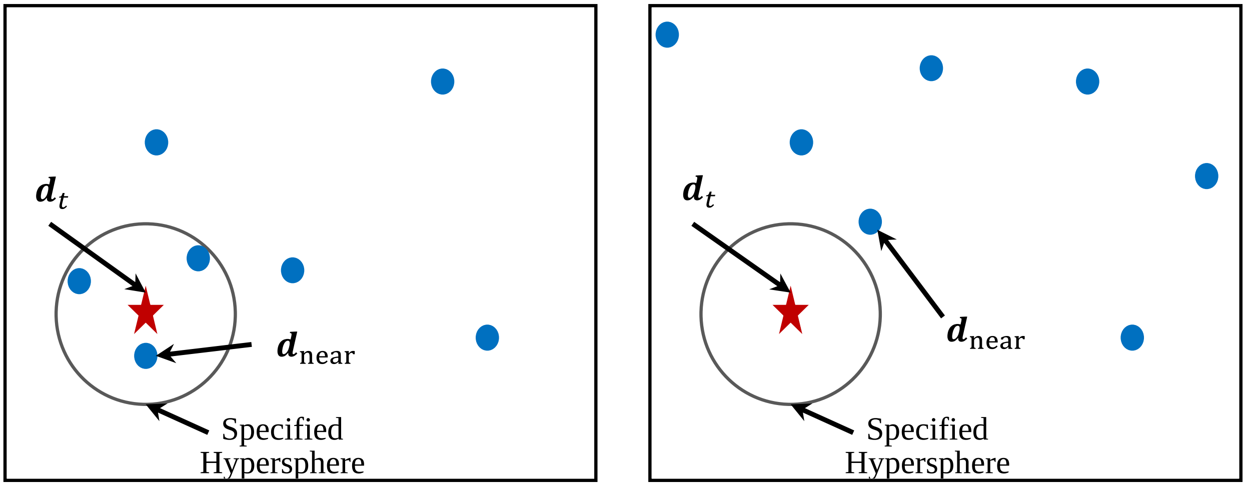
III. RBDO history as an information source

The designs visited in past RBDO iterations already have corresponding adaptively refined surrogates available and these can be stored in a database. The nearest design $\boldsymbol{d}_{\mathrm{near}}$ from past iterations $\{1, \ldots, t-1\}$ will likely have similar failure boundaries as the current design $\boldsymbol{d}_t$. We propose reusing the surrogate $\widehat{g}_{\mathrm{near}}$ corresponding to $\boldsymbol{d}_{\mathrm{near}}$ as an additional information source to supplement the information from the high-fidelity model during the multifidelity active learning process described in Section IV.

However, the information from the nearest past design is only meaningful if the designs are close to each other. Thus, the information from a past design is only reused if $\boldsymbol{d}_{\mathrm{near}}$ is within a specified hypersphere (defined by radius r) around the current design as shown in Figure 2(a). If $\boldsymbol{d}_{\mathrm{near}}$ is not inside the specified hypersphere (Figure 2(b)) at iteration t, the past RBDO history is not reused and only the high-fidelity model is used to run a single-fidelity adaptive sampling method (in this work, EGRA [13]).

When $\boldsymbol{d}_{\mathrm{near}}$ lies inside the specified hypesphere, the two information sources used for building the surrogate for design $\boldsymbol{d}_t$ through the multifidelity active learning algorithm are:

- information source 0 (IS0): high-fidelity model $g(\boldsymbol{d}_t, Z)$



(a) RBDO history reused

(b) RBDO history not reused

Fig. 2 Illustration in a two-dimensional design space that shows when the RBDO history is reused as an additional information source. The blue dots represent designs visited in past optimization iterations and the red star represents the current design.

- information source 1 (IS1): reuse surrogate model $\widehat{g}_{\text{near}}(0, Z)$ from a past RBDO iteration

The associated cost for each information source is given by $c_{l,t}(Z)$, where in the subscript $l \in \{0, 1\}$ denotes the information source and t denotes the optimization iteration. Note that the information sources are specific to the design $\boldsymbol{d}_t$ and change in every RBDO iteration. In this case, we consider that the information sources are noise-free. For simplicity, we limit the discussion in this work to two information sources but the method can be easily extended to more than two information sources.

IV. Multifidelity active learning reusing RBDO history for locating failure boundary

When information is reused, we have two information sources available as described in Section III. In this section, we describe an active learning method that can leverage the multiple information sources to efficiently build an adaptively refined multifidelity surrogate $\widehat{g}_t$ for locating the failure boundary at each RBDO iteration t. A general overview of the proposed approach is shown in Figure 3. We briefly describe the multifidelity GP surrogate used in this work to combine the different information sources in Section IV.A. Section IV.B describes the multifidelity extension of EGRA (mfEGRA) used in this work to refine the surrogate around the failure boundary.

A. Multifidelity Gaussian process surrogate

We use a multifidelity GP surrogate introduced by Poloczek et al. [19] to fuse the different information sources (in this case, IS0 and IS1) into a single GP surrogate $\widehat{g}_t$ for given design $\boldsymbol{d}_t$ that can simultaneously approximate all the available information sources.

The multifidelity GP surrogate is built by making two modeling choices: (1) a GP approximation to IS0 (high-fidelity information source) denoted by $\widehat{g}_t(0, Z) \sim \text{GP}(\mu_0, \Sigma_0)$, and (2) a GP approximation to the model discrepancy between the high-fidelity model and the low-fidelity reused surrogate model as given by $\delta_1 \sim \text{GP}(\mu_1, \Sigma_1)$. μ_l denotes the mean function and Σ_l denotes the covariance kernel for $l \in \{0, 1\}$.

As shown in Ref. [19], for the surrogate model $\widehat{g}_t(l, Z)$ with $l = 1$ is given by $\text{GP}(\mu, \Sigma)$ with

$$\mu(l, z) = \mu_0(z) + \mu_l(z), \tag{4}$$

$$\Sigma((l, z), (l', z')) = \Sigma_0(z, z') + \mathbb{1}_{l,l'}\Sigma_l(z, z'), \tag{5}$$

where $\mathbb{1}_{l,l'}$ denotes the Kronecker's delta. Once the mean function and the covariance kernels are defined using Equations 4 and 5, we can compute the posterior using standard rules of GP regression. A more detailed description

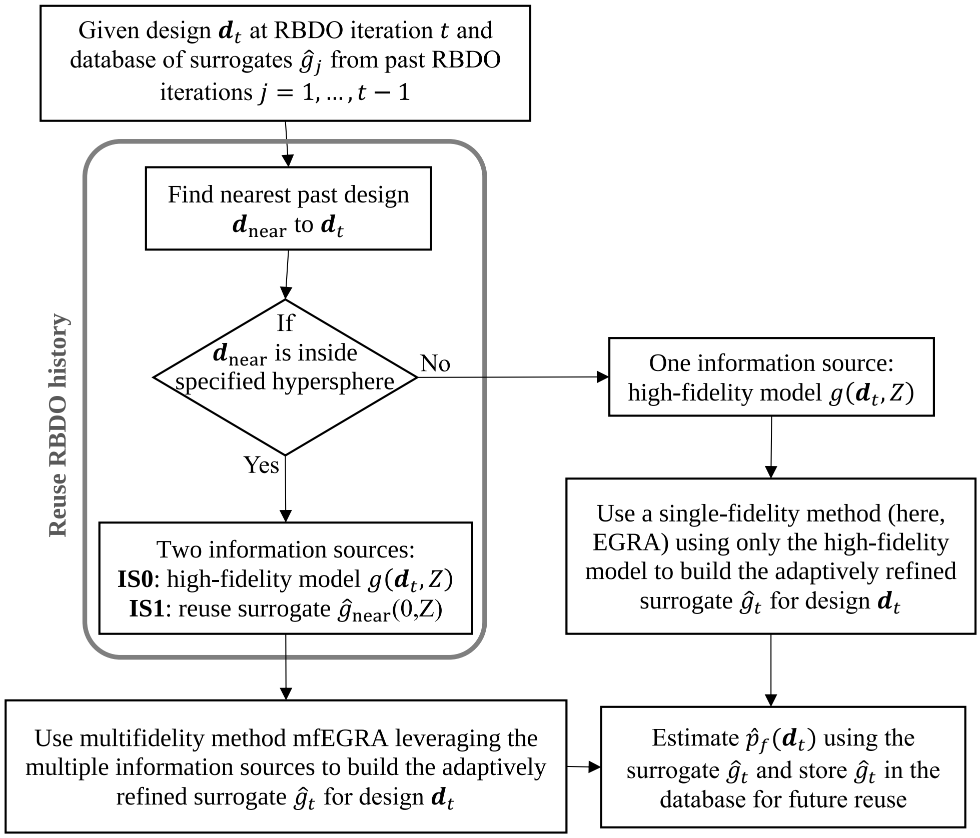


Fig. 3 Overview of the information reuse in multifidelity active learning for RBDO method.

about the assumptions and the implementation of the multifidelity GP surrogate can be found in Ref. [19].

At any given z, the surrogate model posterior distribution of $\widehat{g}_t(l, z)$ is defined by the normal distribution with mean $\mu_t(l, z)$ and variance $\sigma_t^2(l, z) = \Sigma_t((l, z), (l, z))$. The subscript t denotes that the surrogate is built for the given design $\boldsymbol{d}_t$. Consider that n samples $\{[l^i, z^i]\}_{i=1}^n$ have been evaluated for design $\boldsymbol{d}_t$ and these samples are used to fit the present multifidelity GP surrogate. Note that $[l, z]$ is the augmented vector of inputs to the multifidelity GP. The next sample z^{n+1} and the next information source l^{n+1} used to refine the surrogate are found using the two-stage adaptive sampling method described below.

B. Multifidelity extension of EGRA

In this section, we present the active learning strategy mfEGRA, which is a multifidelity extension of EGRA. This strategy enables us to reuse the information from past optimization iterations as an additional information source. mfEGRA picks the location of the next sample and the information source to use for evaluating that sample using a two-stage method that combines the expected feasibility function with an information gain criterion.

1. Location: expected feasibility function

The first stage of mfEGRA involves picking the next location z^{n+1} to be sampled using the expected feasibility function to refine the surrogate in RBDO iteration t. The expected feasibility function, which was used in EGRA [13], defines the expectation of the sample lying within a band around the failure boundary (in this case, $\pm\epsilon(z)$ around the zero contour of the limit state function). For any z, $\mu_t(0, z)$ is the prediction mean and $\sigma_t^2(0, z) = \Sigma_t((0, z), (0, z))$ is the prediction variance provided by the multifidelity GP for the high-fidelity surrogate model. Let $\mathcal{Y}_z \sim \mathcal{N}(\mu_t(0, z), \sigma_t^2(0, z))$ be a normal random variable. Then the feasibility function at any z is defined as being positive within the ϵ-band around the failure boundary and zero otherwise as given by $F(\boldsymbol{d}_t, z) = \epsilon(z) - \min(|\mathcal{Y}_z|, \epsilon(z))$. The expected feasibility function within an ϵ-band around the failure boundary is given by [13]

$$\begin{aligned}\mathbb{E}[F(\boldsymbol{d}_t, z)] =& \mu_t(0,z)\left[2\Phi\left(\frac{-\mu_t(0,z)}{\sigma_t(0,z)}\right) - \Phi\left(\frac{-\epsilon(z)-\mu_t(0,z)}{\sigma_t(0,z)}\right) - \Phi\left(\frac{\epsilon(z)-\mu_t(0,z)}{\sigma_t(0,z)}\right)\right] \\ &- \sigma_t(0,z)\left[2\phi\left(\frac{-\mu_t(0,z)}{\sigma_t(0,z)}\right) - \phi\left(\frac{-\epsilon(z)-\mu_t(0,z)}{\sigma_t(0,z)}\right) - \phi\left(\frac{\epsilon(z)-\mu_t(0,z)}{\sigma_t(0,z)}\right)\right] \\ &+ \epsilon(z)\left[\Phi\left(\frac{\epsilon(z)-\mu_t(0,z)}{\sigma_t(0,z)}\right) - \Phi\left(\frac{-\epsilon(z)-\mu_t(0,z)}{\sigma_t(0,z)}\right)\right],\end{aligned} \tag{6}$$

where Φ is the cumulative distribution function and ϕ is the probability density function of the standard normal distribution. Similar to EGRA [13], we define $\epsilon(z) = 2\sigma_t(0,z)$ to achieve a balance between exploration and exploitation. As mentioned before, we describe the method considering the zero contour as the failure boundary for convenience but the proposed method can be used for any failure boundary by reformulating it as the zero contour.

The location of the next sample is picked by maximizing the expected feasibility function as given by

$$z^{n+1} = \underset{z\in\Omega}{\arg\max}\, \mathbb{E}[F(\boldsymbol{d}_t, z)]. \tag{7}$$

2. Information source: weighted lookahead information gain

The second stage of mfEGRA involves picking the next information source l^{n+1} to be used for simulating the next sample obtained using Equation (7). This is done through a weighted one-step lookahead information gain criterion. This adaptive sampling strategy selects the information source that maximizes the information gain, quantified by the Kullback-Leibler (KL) divergence, in the GP surrogate prediction. We measure the KL divergence between the present surrogate predicted GP and a hypothetical future surrogate predicted GP when a particular information source is used to evaluate the sample at z^{n+1}.

For brevity, we represent the present GP surrogate built using the n available training data by the subscript P as given by $\widehat{g}_P(l,z) = \widehat{g}_t(l, z \mid \{l^i, z^i\}_{i=1}^n)$. The present surrogate predicted Gaussian distribution at any z is given by

$$G_\mathrm{P}(z) \sim \mathcal{N}(\mu_\mathrm{P}(0,z), \sigma^2_\mathrm{P}(0,z)),$$

where $\mu_\mathrm{P}(0,z)$ is the posterior mean and $\sigma^2_\mathrm{P}(0,z)$ is the posterior variance of the present GP surrogate built using the n available training data.

A hypothetical future GP surrogate (denoted by the subscript F) is defined by using a possible future information source $l_F \in \{0,1\}$ to simulate hypothetical data at the sample z^{n+1} as given by

$$G_\mathrm{F}(z|z^{n+1}, l_\mathrm{F}, y_\mathrm{F}) \sim \mathcal{N}(\mu_\mathrm{F}(0,z|z^{n+1}, l_\mathrm{F}, y_\mathrm{F}), \sigma^2_\mathrm{F}(0,z|z^{n+1}, l_\mathrm{F})),$$

where $y_\mathrm{F} \sim \mathcal{N}(\mu_\mathrm{P}(l_\mathrm{F}, z^{n+1}), \sigma^2_\mathrm{P}(l_\mathrm{F}, z^{n+1}))$ is a possible future simulated data at z^{n+1}. The posterior mean of the hypothetical future GP surrogate is $\mu_\mathrm{F}(0,z|z^{n+1}, l_\mathrm{F}, y_\mathrm{F}) \sim \mathcal{N}(\mu_\mathrm{P}(0,z), \bar{\sigma}^2(z|z^{n+1}, l_\mathrm{F}))$, where $\bar{\sigma}^2(z|z^{n+1}, l_\mathrm{F}) = (\Sigma_\mathrm{P}((0,z),(l_\mathrm{F}, z^{n+1})))^2/\Sigma_\mathrm{P}((l_\mathrm{F}, z^{n+1}),(l_\mathrm{F}, z^{n+1}))$ [19]. The posterior variance of the hypothetical future GP surrogate is $\sigma^2_\mathrm{F}(0,z|z^{n+1}, l_\mathrm{F})$ that depends only on the location z^{n+1} and the source l_F. Note that no new evaluations of the information sources are required for the future GP and the total lookahead information gain is obtained by integrating over all possible value of y_F as described below.

Let $D_\mathrm{KL}(G_\mathrm{P}(z) \parallel G_\mathrm{F}(z|z^{n+1}, l_\mathrm{F}, y_\mathrm{F}))$ be the KL divergence between G_P and G_F at any $z \in \Omega$. The total KL divergence can then be calculated by integrating $D_\mathrm{KL}(G_\mathrm{P}(z) \parallel G_\mathrm{F}(z|z^{n+1}, l_\mathrm{F}, y_\mathrm{F}))$ over the entire random variable space Ω as given by

$$\begin{aligned}&\int_\Omega D_\mathrm{KL}(G_\mathrm{P}(z) \parallel G_\mathrm{F}(z|z^{n+1}, l_\mathrm{F}, y_\mathrm{F}))dz \\ &\quad= \int_\Omega \left[log\left(\frac{\sigma_\mathrm{F}(0,z|z^{n+1}, l_\mathrm{F})}{\sigma_\mathrm{P}(0,z)}\right) + \frac{\sigma^2_\mathrm{P}(0,z) + (\mu_\mathrm{P}(0,z) - \mu_\mathrm{F}(0,z|z^{n+1}, l_\mathrm{F}, y_\mathrm{F}))^2}{2\sigma^2_\mathrm{F}(0,z|z^{n+1}, l_\mathrm{F})} - \frac{1}{2}\right] dz.\end{aligned} \tag{8}$$

The total lookahead information gain can then be calculated by taking the expectation of Equation (8) over $y_{\rm F}$ as given by

$$\begin{aligned} D_{\rm IG}(z\| \, z^{n+1}, l_{\rm F}; \boldsymbol{d}_t) &= \mathbb{E}_{y_{\rm F}}\left[\int_{\Omega} D_{\rm KL}(G_{\rm P}(z) \parallel G_{\rm F}(z|z^{n+1}, l_{\rm F}, y_{\rm F}))dz\right] \\ &= \int_{\Omega}\left[log\left(\frac{\sigma_{\rm F}(0, z|z^{n+1}, l_{\rm F})}{\sigma_{\rm P}(0, z)}\right) + \frac{\sigma^2_{\rm P}(0, z) + \bar{\sigma}^2(z|z^{n+1}, l_{\rm F})}{2\sigma^2_{\rm F}(0, z|z^{n+1}, l_{\rm F})} - \frac{1}{2}\right]dz \\ &= \int_{\Omega} D(z|z^{n+1}, l_{\rm F}; \boldsymbol{d}_t)dz \end{aligned} \tag{9}$$

where

$$D(z|z^{n+1}, l_{\rm F}; \boldsymbol{d}_t) = log\left(\frac{\sigma_{\rm F}(0, z|z^{n+1}, l_{\rm F})}{\sigma_{\rm P}(0, z)}\right) + \frac{\sigma^2_{\rm P}(0, z) + \bar{\sigma}^2(z|z^{n+1}, l_{\rm F})}{2\sigma^2_{\rm F}(0, z|z^{n+1}, l_{\rm F})} - \frac{1}{2}.$$

In practice, we choose a discrete set $\mathcal{Z} \subset \Omega$ via Latin hypercube sampling to numerically integrate the information gain given by Equation (9).

A weighted version of the lookahead information gain normalized by the cost of the information source is used to pick the next information source l^{n+1}. The expected feasibility function (Equation (6)) is used to weight the information gain to give more importance to gaining information around the expected failure boundary. The sampling criterion for selecting the next information source l^{n+1} is given by

$$l^{n+1} = \underset{l \in \{0,1\}}{\arg\max} \sum_{z \in \mathcal{Z}} \frac{1}{c_{l,t}(z)} \mathbb{E}[F(\boldsymbol{d}_t, z)] D(z|z^{n+1}, l_{\rm F} = l; \boldsymbol{d}_t). \tag{10}$$

Note that the mfEGRA method reduces to single-fidelity EGRA when there are no extra information sources available, i.e., there are no nearby designs to reuse information from.

The adaptive sampling algorithm continues updating the surrogate for the current design $\boldsymbol{d}_t$ until some stopping criterion is met (in this work, $\mathbb{E}[F(\boldsymbol{d}_t, z)] <= 10^{-10}$). The final adaptively refined surrogate $\widehat{g}_t$ is used for estimating the probability of failure $\widehat{p}_{\rm F}(\boldsymbol{d}_t)$ and stored in the database for future RBDO iterations.

V. Results

We use the acoustic horn problem [20, 21] to demonstrate the effectiveness of reusing surrogates from past optimization iterations as an extra information source in mfEGRA for efficient refinement of surrogates in RBDO. The acoustic horn model used in this work has been used in the context of robust optimization by Ng et al. [20] The inputs to the system are the three random variables listed in Table 1 and the six design variables listed in Table 2. An illustration of the acoustic horn is shown in Figure 4.

Table 1 Uncertain random variables used in the acoustic horn application.

Random variable	Description	Distribution	Lower bound	Upper bound	Mean	Standard deviation
k	wave number	Truncated Normal	1.3	1.5	1.4	0.025
Z_u	upper horn wall impedance	Normal	–	–	50	3
Z_l	lower horn wall impedance	Normal	–	–	50	3

We use a two-dimensional acoustic horn model governed by the non-dimensional Helmholtz equation. In this case, a finite element model of the Helmholtz equation [21] is the high-fidelity model. The model takes on an average 0.3 seconds to run and the cost is taken to be constant. A more detailed description of the acoustic horn model used in this work can be found in Ref. [20]. The output of the model is the reflection coefficient s, which is a measure of the horn's efficiency. We define the failure of the system to be $s(\boldsymbol{d}, z) > 0.1$. The limit state function is defined as $g(\boldsymbol{d}, z) = s(\boldsymbol{d}, z) - 0.1$ so that the failure boundary is defined by $g(\boldsymbol{d}, z) = 0$. In this case, we use a deterministic objective function defined as the reflection coefficient at the mean of the random variables given by

Table 2 Design variables used in the acoustic horn application.

Design variable	Lower bound	Upper bound	Initial design	Best design
b_1	0.679	1.04	0.8595	0.799
b_2	1.04	1.39	1.215	1.196
b_3	1.39	1.75	1.57	1.611
b_4	1.75	2.11	1.93	1.897
b_5	2.11	2.46	2.285	2.310
b_6	2.46	2.82	2.64	2.668

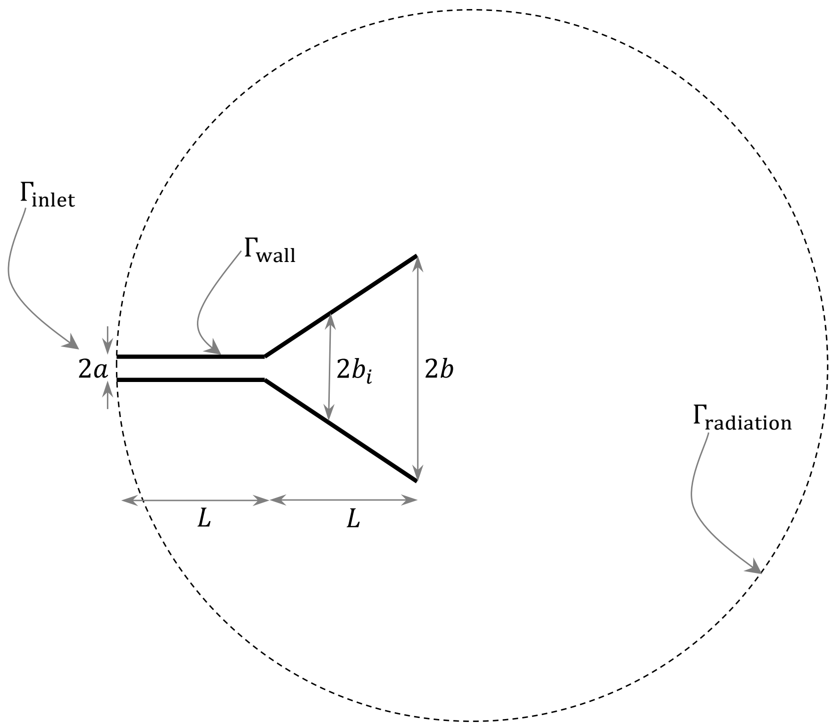


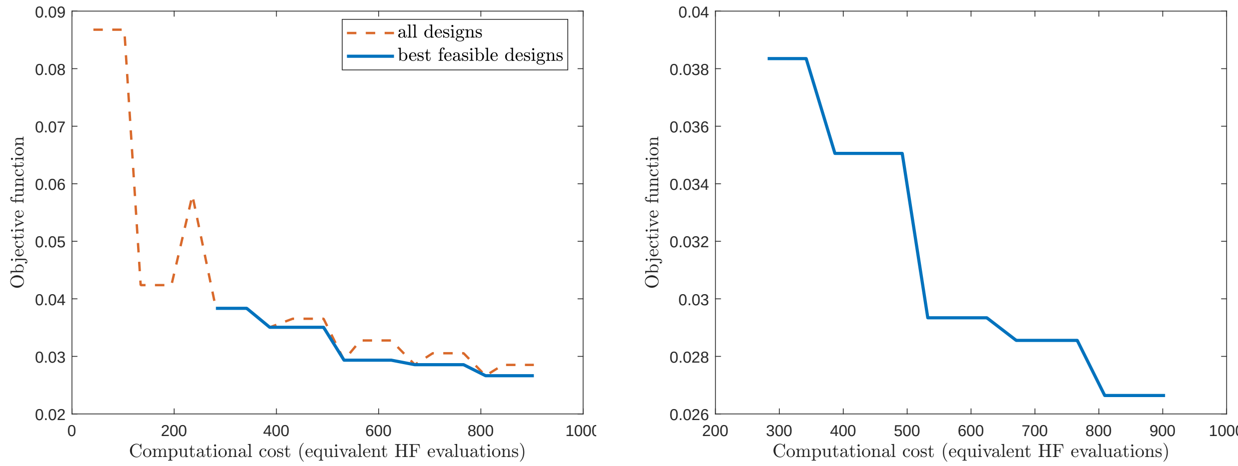
Fig. 4 Two-dimensional acoustic horn geometry with $a = 0.5, b = 3, L = 5$ and shape of the horn flare described by the equally-spaced half-widths $b_i, i = 1, \ldots, 6$. [20]

$s(\boldsymbol{d} \mid k = 1.4, Z_u = 50, Z_l = 50)$. The acoustic horn RBDO problem formulation is given by

$$\begin{aligned} \min_{\boldsymbol{d} \in \mathcal{D}} \quad & s(\boldsymbol{d} \mid k = 1.4, Z_u = 50, Z_l = 50) \\ \text{subject to} \quad & \mathbb{P}(g(\boldsymbol{d}, Z) > 0) \leq 0.006. \end{aligned} \tag{11}$$

We use the same initial design of experiment (DOE) of 10 samples for every RBDO iteration. Latin hypercube sampling [22] is used to create the initial DOE. The mfEGRA algorithm is stopped when the EFF value goes below 10^{-10}. The final GP surrogate in each RBDO iteration is used to estimate the probability of failure and stored for future reuse.

The convergence of the objective function as a function of computational cost (Figure 5) shows that initially we obtain a rapid decrease in the objective function value, but feasibility is only achieved after 300 units of computational cost. This can also be seen from the progress of the optimization from the infeasible to feasible region for the RBDO problem as shown by the probability of failure value for the design in each optimization iteration in Figure 6. It takes 16 optimization iterations to find the first feasible design. However, the optimizer appropriately balances minimizing the objective function while trying to find the feasible region in the design space as indicated by the convergence plot in Figure 5.



(a) (b)

Fig. 5 (a) Objective function value for designs from each optimization iteration, and (b) convergence history of feasible designs for RBDO of the acoustic horn problem using mfEGRA with information reuse.

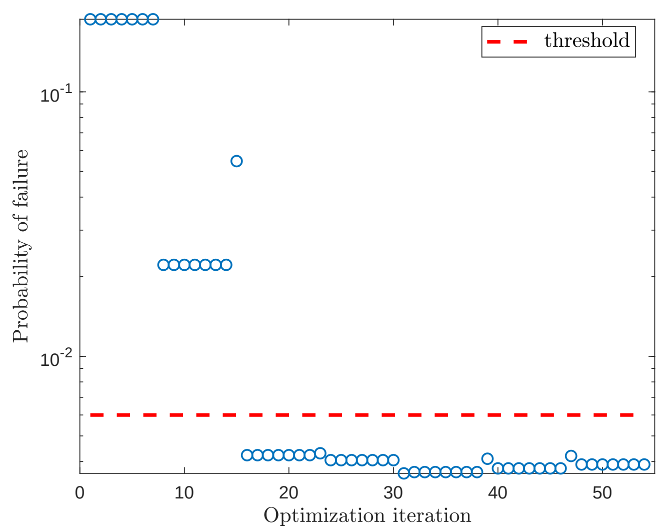


Fig. 6 Probability of failure history in each RBDO iteration for the acoustic horn problem using mfEGRA with information reuse.

Figure 7 shows the computational cost for building the adaptive surrogates using the proposed mfEGRA with information-reuse method as compared to using EGRA with only the high-fidelity model at each optimization iteration. We observe that the proposed mfEGRA method performs as well as or better than single-fidelity EGRA in all optimization iterations. The computational cost of mfEGRA is equal to single-fidelity EGRA when there is no nearby designs because mfEGRA reduces to single-fidelity EGRA without any information reuse. Otherwise, information-reuse-based mfEGRA always performs better. In this case, we reduce the overall computational cost by almost 60% by building the adaptive surrogates through the proposed information-reuse method.

The computational cost for refining the surrogates through the mfEGRA method is low due to the reuse of extremely cheap surrogates from previous optimization iterations as an information source. The allocation of evaluations for each fidelity in the mfEGRA method in each optimization iteration is shown in Figure 8. The allocation of resources to the cheaper low-fidelity model (reused GP model from RBDO history), which takes around 10^{-4} seconds/evaluation, in mfEGRA leads to substantial reduction in computational cost compared to running EGRA that uses only the high-fidelity

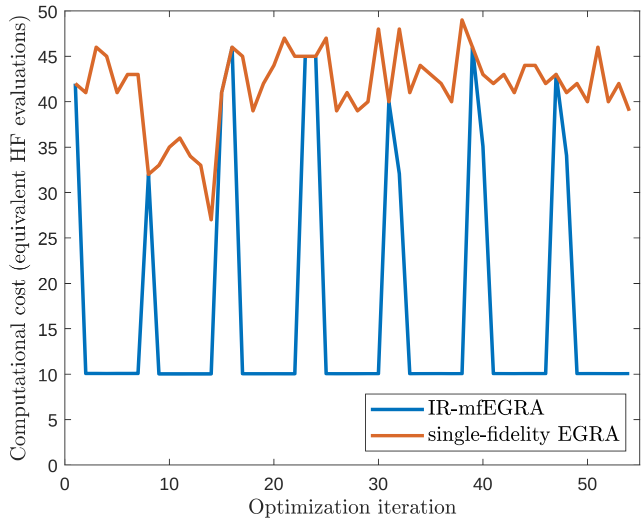


Fig. 7 Comparison of the computational cost for building adaptive surrogates using mfEGRA with information reuse (IR-mfEGRA) vs using single-fidelity EGRA at each optimization iteration.

model, which takes around 0.3 seconds/evaluation.

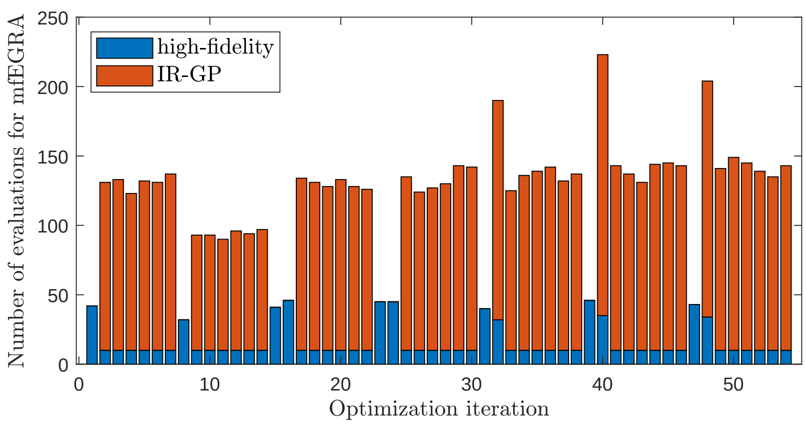


Fig. 8 Number of evaluations required by mfEGRA from the high-fidelity model and the reused GP model (IR-GP) in each RBDO iteration for the acoustic horn problem.

VI. Concluding remarks

This paper introduces a method for reusing information from past RBDO iterations as an additional information source for efficiently refining surrogates around the failure boundary through multifidelity active learning. The information reuse is implemented by reusing the surrogate from the closest design from past RBDO iterations. Reusing the cheap-to-evaluate surrogate as an information source offers substantial reduction in computational cost while refining the current surrogate around the failure boundary. The different information sources are leveraged for adaptively refining the surrogate around the failure boundary by using a multifidelity extension of EGRA. The multifidelity information-reuse method leads to almost 60% reduction in computational cost compared to using EGRA with only the high-fidelity model for building adaptive surrogates during RBDO of the acoustic horn problem.

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